Published by Institute of Physics Publishing for SISSA

Received: *January 22, 2006*
Accepted: *February 9, 2006*
Published: *February 20, 2006*

A c-function for non-supersymmetric attractors

Kevin Goldstein, Rudra P. Jena, Gautam Mandal and Sandip P. Trivedi

Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai, 400 005, India
E-mail: kevin@theory.tifr.res.in, rpjena@theory.tifr.res.in, mandal@theory.tifr.res.in, sandip@theory.tifr.res.in

Abstract: We present a c-function for spherically symmetric, static and asymptotically flat solutions in theories of four-dimensional gravity coupled to gauge fields and moduli. The c-function is valid for both extremal and non-extremal black holes. It monotonically decreases from infinity and in the static region acquires its minimum value at the horizon, where it equals the entropy of the black hole. Higher dimensional cases, involving p-form gauge fields, and other generalisations are also discussed.

Keywords: Black Holes, Black Holes in String Theory, p-branes.

JHEP02(2006)053

Contents

1. Introduction

The attractor phenomenon for extremal black holes has been the subject of considerable investigation. For BPS black holes in $\mathcal{N} = 2$ theories this phenomenon was first studied in [1] and thereafter discussed in [2 – 10]. It has received further attention recently due to the conjecture of [11] and related developments [12 – 16]. For non-supersymmetric extremal black holes, some aspects of the attractor phenomenon were discussed in [7] and [8]. More recently this has been investigated in [17] and [18 – 22]. For important related work see [23, 24].

In supersymmetric black holes, the central charge, which is a function of the moduli and the charges carried by the black hole, plays an important role in the discussion of the attractor. The attractor values of the scalars, which are obtained at the horizon of the black hole, are given by minimising the central charge with respect to the moduli. In the non-supersymmetric case one constructs an effective potential which is a function of the moduli and charges. The attractor values are then given by minimising this effective potential with respect to the moduli.

There is a another sense in which the central charge is also minimised at the horizon of a supersymmetric attractor. One finds that the central charge, now regarded as a function of the position coordinate, evolves monotonically from asymptotic infinity to the horizon and obtains its minimum value at the horizon of the black hole. It is natural to ask whether there is an analogous quantity in the non-supersymmetric case and in particular if the effective potential is also monotonic and minimised in this sense for non-supersymmetric attractors.

JHEP02(2006)053

This paper addresses this question. We present a c-function for non-supersymmetric attractors here. We first study the four dimensional case. The c-function has a simple geometrical and physical interpretation in this case. We are interested in spherically symmetric and static configurations in which all fields are functions of only one variable - the radial coordinate. The c-function, $c(r)$, is given by

$$c(r) = \frac{1}{4}A(r), \tag{1.1}$$

where $A(r)$ is the area of the two-sphere, of the SO(3) isometry group orbit, as a function of the radial coordinate.[1] For any asymptotically flat solution we show that the area function satisfies a c-theorem and monotonically decreases as one moves inwards from infinity. For a black hole solution, the static region ends at the horizon, so in the static region the c-function attains its minimum value at the horizon. This horizon value of the c-function equals the entropy of the black hole. While the horizon value of the c-function is also proportional to the minimum value of the effective potential, more generally, away from the horizon, the two are different. In fact we find that the effective potential need not vary monotonically in a non-supersymmetric attractor. The c-theorem we prove is applicable for supersymmetric black holes as well. In the supersymmetric case, there are three quantities of interest, the c-function, the effective potential and the square of the central charge. At the horizon these are all equal, up to a constant of proportionality. But away from the horizon they are in general different.

We work directly with the second order equations of motion in our analysis and it might seem puzzling at first that one can prove a c-theorem at all. The answer to the puzzle lies in boundary conditions. For black hole solutions we require that the solutions are asymptotically flat. This is enough to ensure that going inwards from asymptotic infinity the c-function decreases monotonically. Without imposing any boundary conditions one cannot prove the c-theorem, as one might expect. But one can show that in the absence of singularities, c can have at most one critical point.

While non-supersymmetric attractors were our primary motivation, the c-theorem is in fact valid for all static, spherically symmetric, asymptotically flat, solutions to the equations of motion.[2] For example, the proof applies also to non-extremal black holes. Once again the Area function must decreases monotonically and its minimum value at the horizon is the entropy.

In our discussion we focus on a system consisting of 4-dimensional gravity coupled to gauge fields and moduli. But in fact the results are more general. The c-theorem is valid for any matter fields which satisfy the null energy condition. This says that,

$$T_{\mu\nu}\zeta^{\mu}\zeta^{\nu} \geqslant 0, \tag{1.2}$$

for any null vector ζ. As long as this energy condition is met and we have a static, spherically symmetric solution that is asymptotically flat, the area function monotonically

[1] We have set $G_N = 1$.

[2] Also there the spacetime region under consideration must be singularity free.

JHEP02(2006)053

decreases, moving in from infinity. The importance of the null energy condition in the proof of a c-theorem was recognised in [25].

One can show that the proof of the c-theorem follows in a straightforward manner from the Raychaudhuri equation and the energy condition, eq. (1.2). By considering a congruence of radially infalling null geodesics one can see that the area $A(r)$ must decrease as one moves inwards from asymptotic infinity. Our focus here is on spherically symmetric configurations, but these comments suggest that a similar c-theorem can be devised more generally as well.

In the latter part of this paper we consider generalisations to higher dimensions. We analyse a system of rank q gauge fields and moduli coupled to gravity and once again find a c-function that satisfies a c-theorem. In $D = p + q + 1$ dimensions this system has extremal black brane solutions whose near horizon geometry is $AdS_{p+1} \times S^q$. We show that the c-function is non-increasing from infinity up to the near horizon region. It's minimum value in the $AdS_{p+1} \times S^q$ region agrees with the conformal anomaly in the dual boundary theory for p even. A c-function in AdS space was considered before in [25, 26] and our construction makes important use of the analysis and results contained therein.

In fact, in the higher dimensional case as well, the c-theorem we prove is more general. It applies to all solutions which have a $\mathrm{SO}(q) \times P$ symmetry, where P is the Poincaré group in $p + 1$ dimensions, as long as suitable boundary conditions are imposed. Both asymptotically flat and asymptotically AdS boundary conditions lead to monotonicity. And both extremal and non-extremal black brane solutions are examples which satisfy the conditions for the c-theorem. Also, the c-theorem works for other kinds of matter we well, as long as the null energy condition holds.

This paper is structured as follows. In section 2, we discuss some background material. Section 3, discusses the c-theorem in 4 dimensions and section 4, the higher dimensional case. Three appendices contain important details.

2. Background

We begin with some background related to the discussion of non-supersymmetric attractors.

Consider a theory consisting of four dimensional gravity coupled to U(1) gauge fields and moduli, whose bosonic terms have the form,

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-G}(R - 2g_{ij}(\partial\phi^i)(\partial\phi^j) - f_{ab}(\phi^i)F^a_{\mu\nu}F^{b\ \mu\nu} - \frac{1}{2}\tilde{f}_{ab}(\phi^i)F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}). \quad (2.1)$$

$F^a_{\mu\nu}, a = 0, \ldots N$ are gauge fields. $\phi^i, i = 1, \ldots n$ are scalar fields. The scalars have no potential term but determine the gauge coupling constants. We note that g_{ij} refers to the metric in the moduli space, this is different from the spacetime metric, $G_{\mu\nu}$.

A spherically symmetric space-time metric in $3 + 1$ dimensions takes the form,

$$ds^2 = -a(r)^2dt^2 + a(r)^{-2}dr^2 + b(r)^2d\Omega^2 \quad (2.2)$$

The Bianchi identity and equation of motion for the gauge fields can be solved by a field strength of the form,

$$F^a = f^{ab}(Q_{eb} - \tilde{f}_{bc}Q^c_m)\frac{1}{b^2}dt \wedge dr + Q^a_m sin\theta d\theta \wedge d\phi, \tag{2.3}$$

where Q^a_m, Q_{ea} are constants that determine the magnetic and electric charges carried by the gauge field F^a, and f^{ab} is the inverse of f_{ab}.

The effective potential $V_{\rm eff}$ is then given by,

$$V_{\rm eff}(\phi_i) = f^{ab}(Q_{ea} - \tilde{f}_{ac}Q^c_m)(Q_{eb} - \tilde{f}_{bd}Q^d_m) + f_{ab}Q^a_m Q^b_m. \tag{2.4}$$

For the attractor mechanism it is sufficient that two conditions to be met. First, for fixed charges, as a function of the moduli, $V_{\rm eff}$ must have a critical point. Denoting the critical values for the scalars as $\phi^i = \phi^i_0$ we have,

$$\partial_i V_{\rm eff}(\phi^i_0) = 0. \tag{2.5}$$

Second, the effective potential must be a minimum at this critical point. I.e. the matrix of second derivatives of the potential at the critical point,

$$M_{ij} = \frac{1}{2}\partial_i\partial_j V_{\rm eff}(\phi^i_0) \tag{2.6}$$

should have positive eigenvalues. Schematically we can write,

$$M_{ij} > 0. \tag{2.7}$$

As discussed in [21], it is possible that some eigenvalues of M_{ij} vanish. In this case the leading correction to the effective potential along the zero mode directions should be such that the critical point is a minimum. Thus, an attractor would result if the leading correction is a quartic term, $V_{\rm eff} = V_{\rm eff}(\phi^i_0) + \lambda(\phi - \phi_H)^4$, with $\lambda > 0$ but not if it is a cubic term, $V_{\rm eff} = V_{\rm eff}(\phi^i_0) + \lambda(\phi - \phi_H)^3$.

Once the two conditions mentioned above are met it was argued in [18] that the attractor mechanism works. There is an extremal Reissner Nordstrom black hole solution in the theory, where the black hole carries the charges specified by the parameters, Q^a_m, Q_{ea} and the moduli take the critical values, ϕ_0 at infinity. For small enough deviations at infinity of the moduli from these values, a double-horizon extremal black hole solution continues to exist. In this extremal black hole the scalars take the same fixed values, ϕ_0, at the horizon independent of their values at infinity. The resulting horizon radius is given by,

$$b^2_H = V_{\rm eff}(\phi^i_0) \tag{2.8}$$

and the entropy is

$$S_{BH} = \frac{1}{4}A = \pi b^2_H. \tag{2.9}$$

In $\mathcal{N} = 2$ supersymmetric theory, $V_{\rm eff}$ can be expressed, [7], in terms of a Kahler potential, K and a superpotential, W as,

$$V_{\rm eff} = e^K[g^{i\bar{j}}\nabla_i W(\nabla_j W)^* + |W|^2], \tag{2.10}$$

where $\nabla_i W \equiv \partial_i W + \partial_i K W$. The Kahler potential and Superpotential in turn can be expressed in terms of a prepotential F, as,

$$K = -\ln \mathrm{Im}\left(\sum_{a=0}^{N} X^{a*}\partial_a F(X)\right), \tag{2.11}$$

and,

$$W = q_a X^a - p^a \partial_a F, \tag{2.12}$$

respectively. Here, $X^a, a = 0, \ldots N$ are special coordinates to describe the special geometry of the vector multiplet moduli space. And q_a, p^a are the electric and magnetic charges carried by the black hole.[3]

For a BPS black hole, the central charge given by,

$$Z = e^{K/2} W, \tag{2.13}$$

is minimised, i.e., $\nabla_i Z = \partial_i Z + \frac{1}{2}\partial_i K Z = 0$. This condition is equivalent to,

$$\nabla_i W = 0. \tag{2.14}$$

The resulting entropy is given by

$$S_{BH} = \pi e^K |W|^2. \tag{2.15}$$

with the Kahler potential and superpotential evaluated at the attractor values.

3. The c-function in 4 dimensions

3.1 The c-function

The equations of motion which follow from eq. (2.1) take the form,

$$\begin{aligned}
R_{\mu\nu} - 2g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j &= f_{ab}\left(2F^a_{\ \mu\lambda}F^{b\ \lambda}_{\ \nu} - \frac{1}{2}G_{\mu\nu}F^a_{\ \kappa\lambda}F^{b\kappa\lambda}\right) \\
\frac{1}{\sqrt{-G}}\partial_\mu\left(\sqrt{-G}g_{ij}\partial^\mu\phi^j\right) &= \frac{1}{4}\partial_i(f_{ab})F^a_{\ \mu\nu}F^{b\mu\nu} \\
&\quad + \frac{1}{8}\partial_i(\tilde{f}_{ab})F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \\
\partial_\mu\left(\sqrt{-G}(f_{ab}F^{b\mu\nu} + \frac{1}{2}\tilde{f}_{ab}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma})\right) &= 0.
\end{aligned} \tag{3.1}$$

We are interested in static, spherically symmetric solutions to the equations of motion. The metric and gauge fields in such a solution take the form, eq. (2.2), eq. (2.3). We will be interested in asymptotically flat solutions below. For these the radial coordinate r in eq. (2.2) can be chosen so that $r \to \infty$ is the asymptotically flat region.

[3] These can be related to Q_{ea}, Q^a_m, using eq. (2.3).

The scalar fields are a function of the radial coordinate alone, and substituting for the gauge fields from, eq. (2.3), the equation of motion for the scalar fields take the form,

$$\partial_r(a^2b^2g_{ij}\partial_r\phi^j) = \frac{\partial_i V_{\rm eff}}{2b^2}, \tag{3.2}$$

where $V_{\rm eff}$ is defined in eq. (2.4).

The Einstein equation for the rr component takes the form of an "energy constraint",

$$-1 + a^2 b'^2 + \frac{a^{2'}b^{2'}}{2} = \frac{-1}{b^2}(V_{\rm eff}(\phi_i)) + a^2b^2g_{ij}(\partial_r\phi^i)\partial_r\phi^j \tag{3.3}$$

Of particular relevance for the present discussion is the equation obtained for $R_{rr} - \frac{G^{tt}}{G^{rr}}R_{tt}$ component of the Einstein equation. From eq. (3.1), this is,

$$\frac{b(r)''}{b(r)} = -g_{ij}\partial_r\phi^i\partial_r\phi^j. \tag{3.4}$$

Here prime denotes derivative with respect to the radial coordinate r.

Our claim is that the c-function is given by,

$$c = \frac{1}{4}A(r), \tag{3.5}$$

where $A(r)$ is the area of the two-sphere defined by constant t and r,

$$A(r) = \pi b^2(r). \tag{3.6}$$

We show below that in any static, spherically symmetric, asymptotically flat solution, c decreases monotonically as we move inwards along the radial direction from infinity. We assume that the spacetime in the region of interest has no singularities and the scalar fields lie in a singularity free region of moduli space with a metric which is positive, i.e., all eigenvalues of the moduli space metric, g_{ij}, are positive. For a black hole we show that the minimum value of c, in the static region, equals the entropy at the horizon.

To prove monotonicity of c it is enough to prove monotonicity of b. Let us define a coordinate $y = -r$ which increases as we move inwards from the asymptotically flat region. We see from eq. (3.4), since the eigenvalues of $g_{ij} > 0$, that $d^2b/dy^2 \leqslant 0$ and so db/dy must be non-increasing as y increases. Now for an asymptotically flat solution, at infinity as $r \to \infty$, $b(r) \to r$. This means $db/dy = -1$. Since db/dy is non-increasing as y increases this means that for all $y > -\infty$, $db/dy < 0$ and thus b is monotonic. This proves the c-theorem.

3.2 Some comments

A few comments are worth making at this stage.

It is important to emphasise that our proof of the c-theorem applies to any spherically symmetric, static solution which is asymptotically flat. This includes both extremal and non-extremal black holes. The boundary of the static region of spacetime, where the killing vector $\frac{\partial}{\partial t}$ is time-like, is the horizon where $a^2 \to 0$. The c function is monotonically

decreasing in the static region, and obtains its minimum value on the boundary at the horizon. We see that this minimum value of c is the entropy of the black hole. We will comment on what happens to c when one goes inside the horizon towards the end of this section.

For extremal black holes it is worth noting that the c-function is not V_{eff} itself. At the horizon, where c obtains its minimum value, the two are indeed equal (up to a constant of proportionality). This follows from the constraint, eq. (3.3), after noting that at a double horizon where a^2 and $a^{2'}$ both vanish, $V_{\text{eff}}(\phi_0^i) = b_H^2$. But more generally, away from the horizon, c and V_{eff} are different. In particular, we will consider an explicit example in appendix A of a flow from infinity to the horizon where V_{eff} does not evolve monotonically.

In the supersymmetric case it is worth commenting that the c-function discussed above and the square of the central charge agree, up to a proportionality constant, at the horizon of a black hole. But in general, away from the horizon, they are different. For example in a BPS extremal Reissner Nordstrom black hole, obtained by setting the scalars equal to their attractor values at infinity, the central charge is constant, while the Area is infinite asymptotically and monotonically decreases to its minimum at the horizon.

It is also worth commenting that c' can vanish identically only in a Robinson-Bertotti spacetime.[4] If c is constant, b is constant. From, eq. (3.4) then ϕ^i are constant. Thus V_{eff} is extremised. It follows from the other Einstein equations then that $a(r) = r/b$ leading to the Robinson-Bertotti spacetime. From this we learn that a flow from one asymptotically (in the sense that c' and all its derivatives vanish) $AdS_2 \times S^2$ where the scalars are at one critical point of V_{eff} to an asymptotically $AdS_2 \times S^2$ spacetime where the scalars are at another critical point is not possible. Once the scalars begin evolving c' will became negative and cannot return to zero.

The c-theorem discussed above is valid more generally than the specific system consisting of gravity, gauge fields and scalars we have considered here. Consider any four-dimensional theory with gravity coupled to matter which satisfies the null energy condition. By this we mean that the stress-energy satisfies the condition,

$$T_{\mu\nu}\zeta^\mu\zeta^\nu \geqslant 0, \tag{3.7}$$

where ζ^a is an arbitrary null vector. One can show that in such a system the c-theorem is valid for all static, spherically symmetric, asymptotically flat, solutions of the equations of motion. To see this, note that from the metric eq. (2.2), it follows that,

$$-R_{tt}G^{tt} + R_{rr}G^{rr} = -2a^2\frac{b''}{b}. \tag{3.8}$$

From Einstein's equations and the null energy condition we learn that the l.h.s above is positive, since

$$-R_{tt}G^{tt} + R_{rr}G^{rr} = T_{\mu\nu}\zeta^\mu\zeta^\nu > 0 \tag{3.9}$$

where $\zeta^\mu = (\zeta^t, \zeta^r)$ are components of a null vector, satisfying the relations, $(\zeta^t)^2 = -G^{tt}, (\zeta^r)^2 = G^{rr}$. Thus as long as we are outside the horizon, and $a^2 > 0$, i.e. in any

[4] By c' vanishing identically we mean that c' and all its derivatives vanish in some region of spacetime.

region of space-time where the Killing vector related to time translations is time-like, $b'' < 0$[5] This is enough to then prove the monotonicity of b and thus c. The importance of the null energy condition for a c-theorem was emphasised in [25].[6]

In fact the c-theorem follows simply from the Raychaudhuri equation and the null energy condition. Consider a congruence of null geodesics, where each geodesic has (θ, ϕ) coordinates fixed, with, (t, r), being functions of the affine parameter, λ. The expansion parameter of this congruence is

$$\vartheta = \frac{d \ln A}{d\lambda}, \tag{3.10}$$

where A is the area, eq. (3.6). Choosing in going null geodesics for which $dr/d\lambda < 0$ we see that $\vartheta < 0$ at $r \to \infty$, for an asymptotically flat space-time. Now, Raychaudhuri's equation tells us that $\frac{d\vartheta}{d\lambda} < 0$ if the energy condition, eq. (3.7), is met. Then it follows that $\vartheta < 0$ for all $r < \infty$ and thus the area A must monotonically decrease. The comments in this paragraph provides a more coordinate independent proof of the c-theorem. Although the focus of this paper is time independent, spherically symmetric configurations, these comments also suggest that a similar c-theorem might be valid more generally. The connection between c-theorems and the Raychaudhuri equation was emphasised in [28, 29].

In the higher dimensional discussion which follows we will see that the c function is directly expressed in terms of the expansion parameter ϑ for radial null geodesics. The reader might wonder why we have not considered an analogous c function in four-dimensions. From the discussion of the previous paragraph we see that any function of the form, $1/\vartheta^p$, where p is a positive power, is monotonically increasing in r. However, in an $AdS_2 \times S^2$ spacetime, $\vartheta \to 0$ and thus such a function will blow up and not equal the entropy of the corresponding extremal black hole.

It seems puzzling at first that a c-function could arise from the analysis of second order equations of motion. As mentioned in the introduction, the answer to this puzzle lies in the fact that we were considering solutions which satisfy asymptotically flat boundary conditions. Without imposing any boundary conditions, we cannot prove monotonicity of c. But one can use the arguments above to show that there is at most one critical point of c as long as the region of spacetime under consideration has no spacetime singularities and also the scalar fields take non-singular values in moduli space. If the critical point occurs at $r = r_*$, c monotonically decreases for all $r < r_*$ and cannot have another critical point. Similarly, for $r > r_*$. From the Raychaudhuri equation it follows that the critical point, at r_*, is a maximum.

Usually the discussion of supersymmetric attractors involves the regions from the horizon to asymptotic infinity. But we can also ask what happens if we go inside the horizon. This is particularly interesting in the non-extremal case where the inside is a time dependent cosmology. In the supersymmetric case one finds that the central charge (and its square) has a minimum at the horizon and increases as one goes away from it towards the

[5] In fact the same conclusion also holds inside the horizon. Now t is space-like and r time-like and $T_{\mu\nu}\zeta^\mu\zeta^\nu = 2a^2\frac{b''}{b} \geqslant 0$. Since $a^2 < 0$, we conclude that $b'' < 0$. We will return to this point at the end of the section.

[6] In [25] this condition is referred to as the weaker energy condition.

outside and also towards the inside. This can be seen as follows. Using continuity at the horizon a modulus take the form in an attractor solution,

$$\phi(r) - \phi_0 \sim |r - r_H|^\alpha \tag{3.11}$$

where α is a positive coefficient and ϕ_0 is the attractor value for the modulus.[7] Since the central charge is minimised by ϕ_0, one finds by expanding in the vicinity of $r = r_H$, that the central charge is also minimised as a function of r.[8] In contrast, the c-function we have considered here, monotonically decreases inside the horizon till we reach the singularity. In fact it follows from the Raychaudhuri equation that the expansion parameter ϑ monotonically decreases and becomes $-\infty$ at the singularity.

4. The c-function in higher dimensions

We analyse higher dimensional generalisations in this section. Consider a system consisting of gravity, gauge fields with rank q field strengths, $F^a_{m_1\cdots m_q}, a = 1, \ldots N$, and moduli $\phi^i, i = 1, \ldots n$, in $p + q + 1$ dimensions, with action,

$$S = \frac{1}{\kappa^2}\int d^D x\sqrt{-G}\left(R - 2g_{ij}(\partial\phi^i)\partial\phi^j - f_{ab}(\phi^i)\frac{1}{q!}F^a_{\mu\nu\ldots}F^{b\ \mu\nu\ldots}\right). \tag{4.1}$$

Take a metric and field strengths of form,

$$ds^2 = a(r)^2\left(-dt^2 + \sum_{i=1}^{p-1} dy_i^2\right) + a(r)^{-2}dr^2 + b(r)^2 d\Omega_q^2, \tag{4.2}$$

$$F^a = Q^a_m \omega_q. \tag{4.3}$$

Here $d\Omega_q^2$ and ω_q are the volume element and volume form of a unit q dimensional sphere sphere. Note that the metric has Poincaré invariance in p direction, t, y_i, and has SO(q) rotational symmetry. The field strengths thread the q sphere and the configuration carries magnetic charge. Other generalisations, which we do not discuss here include, forms of different rank, and also field strengths carrying both electric and magnetic charge.

Define an effective potential,

$$V_{\text{eff}} = f_{ab}(\phi^i)Q^a_m Q^b_m. \tag{4.4}$$

Now, as we discuss further in appendix C, it is easy to see that if V_{eff} has a critical point where $\partial_{\phi^i} V_{\text{eff}}$ vanishes, then by setting the scalars to be at their critical values, $\phi^i = \phi^i_0$, one has extremal and non extremal black brane solutions in this system with

[7] We are working in the coordinates, eq. (2.2). These breakdown at the horizon but are valid for $r > r_H$ and also $r < r_H$ (where $a^2 < 0$). The solution written here is valid in both these regions; for $r = r_H$ we need to take the limiting value.

[8] The effective potential V_{eff} in the non-supersymmetric case is similar. As a function of r it attains a local minimum at the horizon.

metric, eq. (C.13). For extremal solutions, the near horizon limit is $AdS_{p+1} \times S^q$, with metric given by eq. (C.17),

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dy_i^2\right) + \frac{R^2}{r^2}dr^2 + b_H^2 d\Omega_q^2 \tag{4.5}$$

where

$$R = \left(\frac{p}{q-1}\right) b_H \tag{4.6}$$

$$(b_H)^{2(q-1)} = \frac{p}{(p+q-1)(q-1)} V_{\text{eff}}(\phi_0^i). \tag{4.7}$$

In the extremal case, using arguments analogous to [18] one can show that the $AdS_{p+1} \times S^q$ solution is an attractor if the effective potential is minimised at the critical point ϕ_0^i. That is, for small deviations from the attractor values for the moduli at infinity, there is an extremal solution in which the moduli are drawn to their critical values at the horizon and the geometry in the near-horizon region is $AdS_{p+1} \times S^q$.

We now turn to discussing the c-function in this system. The discussion is motivated by the analysis in [25] of a c-theorem in AdS space. Our claim is that a c-function for the system under consideration is given by,

$$c = c_0 \frac{1}{\tilde{A}^{(p-1)}}. \tag{4.8}$$

Here, c_0 is a constant of proportionality chosen so that $c > 0$. $\tilde{A}$ is defined by

$$\tilde{A} = A'\left(\frac{a}{b^{\frac{q}{p-1}}}\right) \tag{4.9}$$

where A is defined to be,

$$A = \ln(a b^{\frac{q}{p-1}}), \tag{4.10}$$

and prime denotes derivative with respect to r. We show below that for any static, asymptotically flat solution of the form, eq. (4.2), c, eq. (4.8), is a monotonic function of the radial coordinate.

The key is once again to use the null energy condition. Consider the $R_{tt}G^{tt} - R_{rr}G^{rr}$ component of the Einstein equation. For the metric, eq. (4.2), we get,

$$-R_{tt}G^{tt} + R_{rr}G^{rr} = a^2\left[-(p-1)\frac{a''}{a} - q\frac{b''}{b}\right] = T_{\mu\nu}\zeta^\mu\zeta^\nu, \tag{4.11}$$

where (ζ^t, ζ^r) are the components of a null vector which satisfy the relation, $(\zeta^t)^2 = -G^{tt}, (\zeta^r)^2 = G^{rr}$. The null energy condition tells us that the r.h.s cannot be negative. For the system under consideration the r.h.s can be calculated giving,

$$-(p-1)\frac{a''}{a} - q\frac{b''}{b} = 2g_{ij}\partial_r\phi^i\partial_r\phi^j. \tag{4.12}$$

It is indeed positive, as would be expected since the matter fields we include satisfy the null energy condition.

From eq. (4.12) we find that

$$\frac{d\tilde{A}}{dr} = -\frac{a}{b^{\frac{q}{p-1}}}\left[\frac{2}{p-1}g_{ij}\phi^i\phi^j + \left(\frac{q}{p-1} + \frac{q^2}{(p-1)^2}\right)\left(\frac{b'}{b}\right)^2\right], \tag{4.13}$$

and thus, $\frac{d\tilde{A}}{dr} \leqslant 0$.

Now we turn to the monotonicity of c. Consider a solution which becomes asymptotically flat as $r \to \infty$. Then, $a \to 1, b \to r$, as $r \to \infty$. It follows then that $\tilde{A} \to 0^+$ asymptotically. Since, we learn from eq. (4.13) that $\tilde{A}$ is a non-increasing function of r it then follows that for all $r < \infty$, $\tilde{A} > 0$. Since, $a, b > 0$, we then also learn from, eq. (4.9), that $A' > 0$ for all finite r.

Next choose a coordinate $y = -r$ which increases as we go in from asymptotic infinity. We have just learned that $dA/dy = -A' < 0$, for finite r. It is now easy to see that

$$\frac{dc}{dy} = -(p-1)\frac{a}{b^{\frac{q}{p-1}}}c\frac{dA}{dy}\frac{1}{\tilde{A}^2}\frac{d\tilde{A}}{dr}. \tag{4.14}$$

Then given that $a, b > 0$, $c > 0$, and $dA/dy < 0$, $\frac{d\tilde{A}}{dr} \leqslant 0$, it follows that $dc/dy \leqslant 0$, so that the c-function is a non-increasing function along the direction of increasing y. This completes our proof of the c-theorem.

For a black brane solution the static region of spacetime ends at a horizon, where a^2 vanishes. The c-function monotonically decreases from infinity and in the static region obtains its minimum value at the horizon. For the extremal black brane the near horizon geometry is $AdS_{p+1} \times S^q$. We now verify that for p even the c function evaluated in the $AdS_{p+1} \times S^q$ geometry agrees with the conformal anomaly in the boundary Conformal Field Theory. From eq. (4.5) we see that in $AdS_{p+1} \times S^q$,

$$a' = 1/R \tag{4.15}$$

$$b = \frac{q-1}{p}R. \tag{4.16}$$

where R is the radius of the AdS_{p+1}. Then

$$c \propto \frac{R^{p+q-1}}{G_N^{p+q+1}} \propto \frac{R^{p-1}}{G_N^{p+1}} \tag{4.17}$$

where G_N^{p+q+1}, G_N^{p+1} refer to Newton's constant in the $p+q+1$ dimensional spacetime and the $p+1$ dimensional spacetime obtained after KK reduction on the S^q respectively. The right hand side in eq. (4.17) is indeed proportional to the value of the conformal anomaly in the boundary theory when p is even [30]. By choosing c_0, eq. (4.8), appropriately, they can be made equal. Let us also comment that c in the near horizon region can be expressed in terms of the minimum value of the effective potential. One finds that $c \propto (V_{\rm eff}(\phi_0^i))^{\frac{(p+q-1)}{2(q-1)}}$, where the critical values for the moduli are $\phi^i = \phi_0^i$.

A few comments are worth making at this stage. We have only considered asymptotically flat spacetimes here. But our proof of the c-theorem holds for other cases as well. Of

particular interest are asymptotically $AdS_{p+1} \times S^q$ spacetime. The metric in this case takes the form, eq. (4.5), as $r \to \infty$. The proof is very similar to the asymptotically flat case. Once again one can argue that $A' > 0$ for $r < \infty$ and then defining a coordinate $y = -r$ it follows that dc/dy is a non-increasing function of y. The c-theorem allows for flows which terminate in another asymptotic $AdS_{p+1} \times S^q$ spacetime. The second $AdS_{p+1} \times S^q$ space-time, which lies at larger y, must have smaller c. Such flows can arise if V_{eff} has more than one critical point. It is also worth commenting that requiring that c is a constant in some region of spacetime leads to the unique solution (subject to the conditions of a metric which satisfies the ansatz, eq. (4.2)) of $AdS_{p+1} \times S^q$ with the scalars being constant and equal to a critical value of V_{eff}.

We mentioned above that our definition of the c function is motivated by [25]. Let us make the connection clearer. The c-function in[9] [25, 26] is defined for a spacetime of the form,

$$ds^2 = e^{2A} \sum_{\mu,\nu=0,\ldots p} \eta_{\mu\nu} dy^\mu dy^\nu + dz^2, \tag{4.18}$$

and is given by

$$c = \frac{c_0}{(dA/dz)^{p-1}}. \tag{4.19}$$

Note that eq. (4.18) is the Einstein frame metric in $p+1$ dimensions. Starting with the metric, eq. (4.2), and Kaluza-Klein reducing over the Q sphere shows that A defined in eq. (4.10) agrees with the definition eq. (4.18) above and dA/dz agrees with $\tilde{A}$ in eq. (4.9). This shows that the c-function eq. (4.8) and eq. (4.19)are the same.

The monotonicity of c follows from that of $\tilde{A}$, eq. (4.9). One can show that for a congruence of null geodesics moving in the radial direction, with constant (θ, ϕ), the expansion parameter ϑ is given by,

$$\vartheta = \left(\frac{a'}{a} + \frac{q}{p-1} \frac{b'}{b} \right). \tag{4.20}$$

Raychaudhuri's equation and the null energy condition then tells us that $\frac{d\vartheta}{dr} < 0$. However, in an $AdS_{p+1} \times S^q$ spacetime ϑ diverges, this behaviour is not appropriate for a c-function. From eq. (4.9) we see that $\tilde{A}$ differs from ϑ by an additional multiplicative factor, $a/b^{\frac{q}{p-1}}$. This factor is chosen to preserve monotonicity and now ensures that c goes to a finite constant in $AdS_{p+1} \times S^q$ spacetimes. A similar comment also applies to the c-function discussed in [25].

5. Concluding comments

In two-dimensional field theories it has been suggested sometime ago [31 – 33] that the c function plays the role of a potential, so that the RG equations take the form of a gradient flow,

$$\beta_i = -\frac{\partial c}{\partial g^i},$$

[9] Another c-function has been defined in [27].

where c is the Zamolodchikov c-function [34]. This phenomenon has a close analogy in the case of supersymmetric black holes, where the radial evolution of the moduli is determined by the gradient of the central charge in a first order equation. In contrast, the c-function we propose does not satisfy this property in either the supersymmetric or the non-supersymmetric case. In particular, in the non-supersymmetric case the scalar fields satisfy a second order equation and in particular the gradient of the c-function does not directly determine their radial evolution.

It might seem confusing at first that our derivation of the c-theorem followed from the second order equations of motion. The following simple mechanically model is useful in understanding this. Consider a particle moving under the force of gravity. The c-function in this case is the height x which satisfies the condition

$$\ddot{x} = -g, \tag{5.1}$$

where g is the acceleration due to gravity. Now, if the initial conditions are such that $\dot{x} < 0$ then going forwards in time x will monotonically decrease. However, if the direction of time is chosen so that $\dot{x} > 0$, going forward in time there will be a critical point for x and thus x will not be a monotonic function of time. In this case though there can be at most one such critical point.

While the equations of motion that govern radial evolution are second order, the attractor boundary conditions restrict the allowed initial conditions and in effect make the equations first order. This suggests a close analogy between radial evolution and RG flow. The existence of a c-function which we have discussed in this paper adds additional weight to the analogy. In the near-horizon region, where the geometry is $AdS_{p+1} \times S^q$, the relation between radial evolution and RG flow is quite precise and well known. The attractor behaviour in the near horizon region can be viewed from the dual CFT perspective. It corresponds to turning on operators which are irrelevant in the infra-red. These operators are dual to the moduli fields in the bulk, and their being irrelevant in the IR follows from the fact that the mass matrix, eq. (2.6), has only positive eigenvalues.

It is also worth commenting that the attractor phenomenon in the context of black holes is quite different from the usual attractor phenomenon in dynamical systems. In the latter case the attractor phenomenon refers to the fact that there is a universal solution that governs the long time behaviour of the system, regardless of initial conditions. In the black hole context a generic choice of initial conditions at asymptotic infinity does not lead to the attractor phenomenon. Rather there is one well behaved mode near the horizon and choosing an appropriate combination of the two solutions to the second order equations at infinity allows us to match on to this well behaved solution at the horizon. Choosing generic initial conditions at infinity would also lead to triggering the second mode near the horizon which is ill behaved and typically would lead to a singularity.

Finally, we end with some comments about attractors in cosmology. Scalar fields exhibit a late time attractor behaviour in FRW cosmologies with growing scale factor (positive Hubble constant H). Hubble expansion leads to a friction term in the scalar field equations,

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0. \tag{5.2}$$

JHEP02(2006)053

As a result at late times the scalar fields tend to settle down at the minimum of the potential generically without any precise tuning of initial conditions. This is quite different from the attractor behaviour for black holes and more akin to the attractor in dynamical systems mentioned above.

Actually in AdS space there is an analogy to the cosmological attractor. Take a scalar field which has a negative $(mass)^2$ in AdS space (above the BF bound). This field is dual to a relevant operator. Going to the boundary of AdS space a perturbation in such a field will generically die away. This is the analogue of the late time behaviour in cosmology mentioned above. Similarly there is an analogue to the black hole attractor in cosmology. Consider dS space in Poincaré coordinates,

$$ds^2 = -\frac{dt^2}{t^2} + t^2 dx_i^2, \tag{5.3}$$

and a scalar field with potential V propagating in this background. Notice that $t \to 0$ is a double horizon. For the scalar field to be well behaved at the horizon, as $t \to 0$, it must go to a critical point of V, and moreover this critical point will be stable in the sense that small perturbations of the scalar about the critical point will bring it back, if $V'' < 0$ at the critical point, i.e., if the critical point is a maximum. This is the analogue of requiring that $V_{\rm eff}$ is at a minimum for attractor behaviour in black hole.[10] It is amusing to note that a cosmology in which scalars are at the maximum of their potential, early on in the history of the universe, could have other virtues as well in the context of inflation.

Acknowledgments

We thank N. Iizuka for discussion. This research is supported by the Government of India. S.T. acknowledges support from the Swarnajayanti Fellowship, DST, Govt. of India. Most of all we thank the people of India for generously supporting research in String Theory.

A. $V_{\rm eff}$ need not be monotonic

In this appendix we construct an explicit example showing that $V_{\rm eff}$ as a function of the radial coordinate need not be monotonic. The basic point in our example is simple. The scalar field ϕ is a monotonic function of the radial coordinate, r, eq. (2.2) . But the effective potential is not a monotonic function of ϕ, and as a result is not monotonic in r.

We work with the following simple $V_{\rm eff}$ to construct such a solution,

$$V_{\rm eff} = V_{01} + \frac{1}{2}m^2(\phi - \phi_{01})^2, \ \phi \leqslant \phi_a \tag{A.1}$$

$$V_{\rm eff} = V_{02} - \frac{1}{2}m^2(\phi - \phi_{02})^2, \ \phi \geqslant \phi_a. \tag{A.2}$$

At ϕ_a, the potential is continuous, giving the relation,

$$V_{02} = V_{01} + \frac{1}{2}m^2(\phi - \phi_{01})^2 + \frac{1}{2}m^2(\phi - \phi_{02})^2. \tag{A.3}$$

[10] The sign reversal is due to the interchange of a space and time directions.

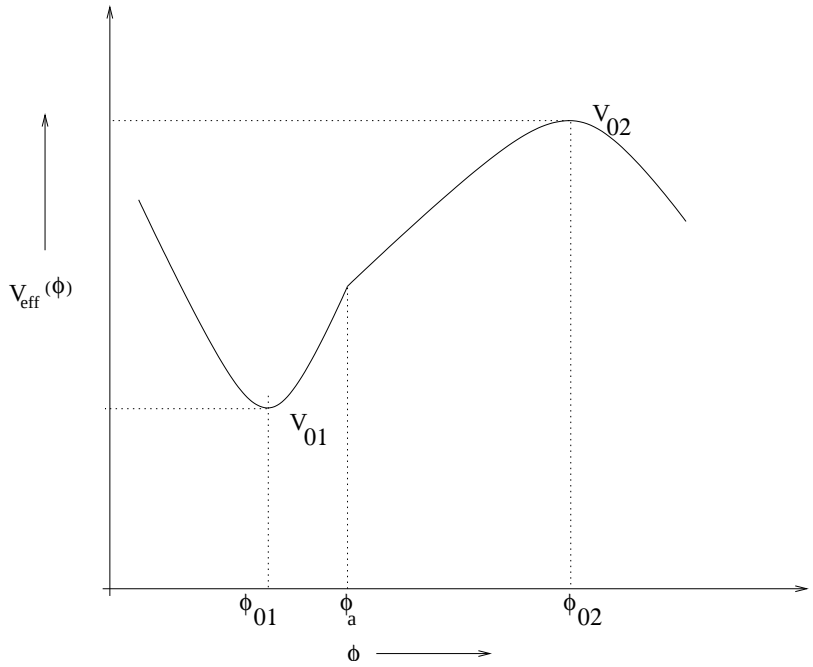

Figure 1: The effective potential $V_{\rm eff}$ as a function of ϕ

We will take the potential as being specified by $V_{01}, \phi_{01}, \phi_{02}, \phi_a, m^2$ with V_{02} being determined by eq. (A.3). The effective potential is given in figure 1. Note that with a minimum at ϕ_{01} and a maximum at ϕ_{02}, $V_{\rm eff}$, is a non-monotonic function of ϕ. Note also that the the first derivative of the potential has a finite jump at $\phi = \phi_a$. Since the equations of motion are second order this means the scalar fields and the metric components, a, b, and their first derivatives will be continuous across ϕ_a. The finite jump is thus mild enough for our purposes.

The attractor value for the scalar is ϕ_{01}. By setting $\phi = \phi_{01}$, independent of r, we get an extremal Reissner Nordstrom black hole solution. The radius of the horizon, r_H in this solution is given by

$$r_H^2 = V_{01}. \tag{A.4}$$

This solution is our starting point. We now construct the solution of interest in perturbation theory, following the analysis in [18], whose conventions we also adopt. For the validity of perturbation theory, we take, $\phi_a - \phi_{01} \ll 1$, and also $\phi_{02} - \phi_{01} \ll 1$. The non-monotonicity of the potential then comes into play even when the scalar field makes only small excursions around the minimum ϕ_{01}. In addition we will also take, $\frac{4m^2}{r_H^2} < 1$, it then follows that $\frac{V_{02}-V_{01}}{V_{01}} \ll 1$.

We construct the solution for the scalar field to first order in perturbation theory below. In the solution the scalar field is a monotonic function of r. This allows the solution to be described in two regions. In region I, $\phi_{01} \leqslant \phi \leqslant \phi_a$, it is given by,

$$\phi = \phi_{01} + A(r - r_H)^\alpha, \tag{A.5}$$

$$\alpha = \frac{1}{2}\left(\sqrt{1+\frac{4m^2}{r_H^2}}-1\right). \tag{A.6}$$

And in region II, $\phi > \phi_a$, it is given by,

$$\phi = \phi_{02} + B_1(r-r_H)^{(-\gamma_1)} + B_2(r-r_H)^{(-\gamma_2)}, \tag{A.7}$$

$$\gamma_1 = \frac{1}{2}\left(1+\sqrt{1-\frac{4m^2}{r_H^2}}\right), \tag{A.8}$$

$$\gamma_2 = \frac{1}{2}\left(1-\sqrt{1-\frac{4m^2}{r_H^2}}\right). \tag{A.9}$$

The boundary between the two is at r_a, where $\phi = \phi_a$, and ϕ and its first derivative with respect to r are continuous. The continuity conditions allow us to solve for B_1, B_2, in terms of A, and also determine r_a in terms of A. The solution is thus completely specified by the constant, A. r_a satisfies the relation,

$$\left(1-\frac{r_h}{r_a}\right)^{\alpha} = \frac{(\phi_a-\phi_{01})}{A}. \tag{A.10}$$

We will omit some details of the subsequent analysis. One finds that as long as

$$(\phi_a-\phi_{01}) < A < \left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{\alpha}{\gamma_1-\gamma_2}}(\phi_a-\phi_{01}), \tag{A.11}$$

the scalar field monotonically evolves with r and transits from region I to region II as r increases. Now we see from eq. (A.7) that if $B_1+B_2>0$, $\phi(r\to\infty) > \phi_{02}$. This ensures that V_{eff} is not a monotonic function of r. It will first increases and then decreases as r decreases from ∞ to r_H. The condition, $B_1+B_2>0$, gives rise to the condition,

$$(\phi_{02}-\phi_a) < \alpha\frac{[1-(1-\frac{r_H}{r_a})^{\gamma_1-\gamma_2}]}{[\gamma_1-\gamma_2(1-\frac{r_H}{r_a})^{\gamma_1-\gamma_2}]}(\phi_a-\phi_{01}). \tag{A.12}$$

Having picked a value of A that lies in the range, eq. (A.11), we can then determine r_a from eq. (A.10). As long as ϕ_{02} is small enough and satisfies condition eq. (A.12) we see that the asymptotic value of $\phi(r\to\infty) > \phi_{02}$. It then follows, as argued above, that in the resulting solution V_{eff} is not a monotonic function of r.

We end with three comments. First, we have not obtained the the corrections to the metric components a, b in perturbation theory here. But this can be done following the analysis in [18]. One finds that the corrections are small. Second, the c-function is of course monotonic as a function of the radial coordinate in this example too. The area of the extremal Reissner Nordstrom black hole monotonically decreases and this is true even after including the small corrections in perturbation theory. Finally, we have not obtained the effective potential above starting with gauge fields coupled to moduli. In fact, for dilaton-like couplings, the simplest example we have been able to construct, where V_{eff} has multicritical points with some minimal and maxima, involves two moduli, a dilaton and axion, and two gauge fields. Our discussion above has a close parallel in this case as well

(with both dilaton and axion excited) and we expect, by dialling the charges and couplings, that the analogue of condition eq. (A.12) can be met leading to solutions where $V_{\rm eff}$ evolves non-monotonically with the radial coordinate r.

B. More details in higher dimensional case

The equations of motion that follow from the action, eq. (4.1), are,

$$\begin{aligned} R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i &= \frac{q}{q!}f_{ab}(\phi_i)F^a_{\mu\lambda....}F^{b\lambda...}_{\nu} - \frac{q-1}{(p+q-1)q!}G_{\mu\nu}f_{ab}(\phi_i)F^a_{\mu\nu....}F^{b\ \mu\nu...} \\ \frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_i) &= \frac{1}{4q!}\partial_i f_{ab}(\phi_i)F^a_{\mu\nu....}F^{b\ \mu\nu...} \\ \partial_\mu(\sqrt{-G}f_{ab}(\phi_i)F^{b\mu\nu}) &= 0. \end{aligned} \tag{B.1}$$

Substituting for the gauge fields from eq. (4.3) we learn that $R_{tt} = \frac{a^2}{b^2}(\frac{q-1}{p})R_{\theta\theta}$, which yields the equation,

$$pb^2\left(pa^{'2} + \frac{qaa^{'}b^{'}}{b} + aa^{''}\right) = (q-1)\left((q-1) - (p+1)aba^{'}b^{'} - a^2\left((q-1)b^{'2} + bb^{''}\right)\right) \tag{B.2}$$

where we have computed the curvature components using the metric, eq. (4.2). The $R_{rr} - \frac{G^{tt}}{G^{rr}}R_{tt}$ component of the Einstein equation gives

$$(p-1)\frac{a^{''}}{a} + \frac{qb^{''}}{b} = -2g_{ij}\partial_r\phi^i\partial_r\phi^j. \tag{B.3}$$

Also the R_{rr} component itself yields a first order "energy" constraint,

$$(p(p-1)b^2a^{'2} + 2pqaba^{'}b^{'} + q(q-1)(-1+a^2b^{'2})) = 2a^2b^2g_{ij}\partial_r\phi^i\partial_r\phi^j - V_{\rm eff}(\phi_i)b^{-2(q-1)} \tag{B.4}$$

where $V_{\rm eff}$ is defined in eq. (4.4).

The equation of motion of the scalar field is given by,

$$\partial_r(a^{p+1}b^q\partial_r g_{ij}\phi^j) = \frac{a^{p-1}\partial_i V_{\rm eff}}{4b^q}. \tag{B.5}$$

Setting $\phi^i = \phi^i_0$, where ϕ^i_0 is a critical point of $V_{\rm eff}$ one finds that $AdS_{p+1}\times S^q$ is a solution of these equations with metric, eq. (4.5).

C. Higher dimensional p-brane solutions

Fixing the scalars at their attractor values, as described in section 4, we are left with the action

$$S = \frac{1}{\kappa^2}\int d^Dx\sqrt{-G}\left\{R - \frac{1}{q!}\sum_a F^{a\ 2}_{(q)}\right\} \tag{C.1}$$

where f_{ab} has been diagonalised and the attractor values of the scalars have been absorbed into the a redefinition of the gauge charges, Q^a. We denote the new charges as $\bar{Q}^a$.

To find solutions, we can dimensionally reduce this action along the brane and use known blackhole solutions. To this end take the metric

$$ds^2 = \underbrace{e^{\lambda\rho} d\hat{s}^2}_{t,r,\omega_1,\ldots,\omega_q} + \underbrace{e^{-\left(\frac{q}{p-1}\right)\lambda\rho} dy^2}_{i_1\ldots i_{p-1}} \tag{C.2}$$

where

$$\lambda = \pm\sqrt{\frac{2(p-1)}{q(p+q-1)}} \tag{C.3}$$

then

$$R = e^{-\lambda\rho}\left(\hat{R} - \lambda^2\hat{\nabla}^2\rho - \frac{1}{2}(\hat{\nabla}\rho)^2\right) \tag{C.4}$$

where $\hat{R}$ and $\hat{\nabla}$ are respectively the Ricci scalar and covariant derivative for $d\hat{s}^2$. The coefficient, λ, has been fixed by requiring that, we remain in the Einstein frame, and that the kinetic term for ρ has canonical normalisation. Upon neglecting the boundary term, the action becomes

$$S = \frac{V_{(p-1)}}{\kappa^2}\int d^{(q+2)}x\sqrt{-\hat{G}}\left\{\hat{R} - \frac{1}{2}\left(\hat{\nabla}\rho\right)^2 - \frac{1}{q!}e^{\beta\rho}\sum_a\left(\hat{F}^a_{(q)}\right)^2\right\} \tag{C.5}$$

where

$$\beta = -(q-1)\lambda. \tag{C.6}$$

The black hole solution to eq. (C.5) is [35, 36]:

$$d\hat{s}^2 = -\left(f_+\right)\left(f_-\right)^{1-\hat{\gamma}(q-1)}dt^2 + \left(f_+\right)\left(f_-\right)^{\hat{\gamma}-1}du^2 + \left(f_-\right)^{\hat{\gamma}}u^2 d\Omega_q^2 \tag{C.7}$$

$$e^{\lambda\rho} = \left(f_-\right)^{-\hat{\gamma}} \tag{C.8}$$

$$f_\pm = \left(1 - \left(\frac{u_\pm}{u}\right)^{q-1}\right) \tag{C.9}$$

where

$$\hat{\gamma} = \frac{2(p-1)}{(q-1)p} \tag{C.10}$$

with

$$\hat{F}^a = \bar{Q}^a\omega_q \tag{C.11}$$

$$\sum_a\left(\bar{Q}^a\right)^2 = \frac{\hat{\gamma}(q-1)^3(u_+u_-)^{q-1}}{\beta^2}. \tag{C.12}$$

Using eq. (C.2) we find the solution to the original action, eq. (C.1), is

$$ds^2 = \left(f_-\right)^{\frac{2}{p}}\left(-\left(\frac{f_+}{f_-}\right)dt^2 + dy^2\right) + \left(f_+f_-\right)^{-1}du^2 + u^2 d\Omega_q^2. \tag{C.13}$$

So finally, the extremal solution is

$$ds^2 = (f)^{\frac{2}{p}}\left(-dt^2 + dy^2\right) + (f)^{-2}\, du^2 + u^2 d\Omega_q^2 \tag{C.14}$$

$$f = \left(1 - \left(\frac{b_H}{u}\right)^{q-1}\right) \tag{C.15}$$

where $b_H = u_{\pm}$. Now we take the near horizon limit,

$$u \xrightarrow{\epsilon \to 0} b_H + \epsilon R \left(\frac{r}{R}\right)^p, \tag{C.16}$$

with t and y rescaled appropriately, which indeed gives the near horizon geometry $AdS_{p+1} \times S^q$:

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dy^2\right) + \frac{R^2}{r^2} dr^2 + b_H^2 d\Omega_q^2 \tag{C.17}$$

where

$$R = \left(\frac{p}{q-1}\right) b_H \tag{C.18}$$

and

$$V_{\text{eff}} \overset{\text{eq.}(C.12)}{=} \frac{(p+q-1)(q-1)}{p} (b_H)^{2(q-1)}. \tag{C.19}$$

References

[1] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[2] M. Cvetič and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, *Phys. Rev.* **D 53** (1996) 5619 [hep-th/9512031].

[3] A. Strominger, *Macroscopic entropy of $N = 2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[4] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136].

[5] S. Ferrara and R. Kallosh, *Universality of supersymmetric attractors*, *Phys. Rev.* **D 54** (1996) 1525 [hep-th/9603090].

[6] M. Cvetič and C.M. Hull, *Black holes and U-duality*, *Nucl. Phys.* **B 480** (1996) 296 [hep-th/9606193].

[7] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [hep-th/9702103].

[8] G.W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges and the first law of black hole thermodynamics*, *Phys. Rev. Lett.* **77** (1996) 4992 [hep-th/9607108].

[9] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049].

[10] F. Denef, B.R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, *JHEP* **05** (2001) 012 [hep-th/0101135].

JHEP02(2006)053

[11] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[12] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [hep-th/9812082].

[13] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [hep-th/9906094].

[14] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [hep-th/0409148].

[15] H. Ooguri, C. Vafa and E.P. Verlinde, *Hartle-Hawking wave-function for flux compactifications*, *Lett. Math. Phys.* **74** (2005) 311 [hep-th/0502211].

[16] R. Dijkgraaf, R. Gopakumar, H. Ooguri and C. Vafa, *Baby universes in string theory*, hep-th/0504221.

[17] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177].

[18] K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, *Non-supersymmetric attractors*, *Phys. Rev.* **D 72** (2005) 124021 [hep-th/0507096].

[19] R. Kallosh, *Flux vacua as supersymmetric attractors*, hep-th/0509112.

[20] R. Kallosh, *New attractors*, *JHEP* **12** (2005) 022 [hep-th/0510024].

[21] P.K. Tripathy and S.P. Trivedi, *Non-supersymmetric attractors in string theory*, hep-th/0511117.

[22] A. Giryavets, *New attractors and area codes*, hep-th/0511215.

[23] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034 [hep-th/0506176].

[24] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022 [hep-th/0508218].

[25] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017].

[26] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, *JHEP* **12** (1998) 022 [hep-th/9810126].

[27] S. Nojiri, S.D. Odintsov and S. Ogushi, *Finite action in D5 gauged supergravity and dilatonic conformal anomaly for dual quantum field theory*, *Phys. Rev.* **D 62** (2000) 124002 [hep-th/0001122].

[28] V. Sahakian, *Holography, a covariant c-function and the geometry of the renormalization group*, *Phys. Rev.* **D 62** (2000) 126011 [hep-th/9910099].

[29] E. Alvarez and C. Gomez, *Holography and the c-theorem*, hep-th/0009203.

[30] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

JHEP02(2006)053

[31] T. Banks and E.J. Martinec, *The renormalization group and string field theory*, *Nucl. Phys.* **B 294** (1987) 733.

[32] S.R. Das, G. Mandal and S.R. Wadia, *Stochastic quantization on two-dimensional theory space and morse theory*, *Mod. Phys. Lett.* **A 4** (1989) 745.

[33] C. Vafa, *c theorem and the topology of 2-d QFTS*, *Phys. Lett.* **B 212** (1988) 28.

[34] A.B. Zamolodchikov, *'Irreversibility' of the flux of the renormalization group in a 2-d field theory*, *JETP Lett.* **43** (1986) 730.

[35] G.W. Gibbons and K.-I. Maeda, *Black holes and membranes in higher dimensional theories with dilaton fields*, *Nucl. Phys.* **B 298** (1988) 741.

[36] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B 360** (1991) 197.

JHEP02(2006)053